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LETTER TO THE EDITOR

# Multicritical points in matrix models

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**Abstract.** The simplest matrix model which exhibits multicritical points is carefully analysed. We reproduce recent results of potential interest for the non-perturbative theory of strings in the region where the orthogonal polynomials were correctly used. However, the present analysis holds for the whole parameter space.

Statistical mechanics models on random lattices are important for the description of many systems of condensed matter and also for purely theoretical reasons. In recent years an important advance was the formulation of the Ising model on lattices given by planar Feynman graphs [1, 2]. The partition function of the model was written in terms of two sets of random matrices and its closed solution in the thermodynamic limit was shown to be the large-$N$ limit of the 'two-matrix model', solved long ago [3, 4].

Remarkably, in this formulation, it is sometimes possible to evaluate analytically some thermodynamic observables which were not obtained in closed form for the corresponding model on a regular, fixed lattice. The $q$-state Potts model [5], the $O(N)$ model [6], the spectra of polymers [7] and the ADE models [8] on Feynman graphs were similarly formulated in terms of random matrix models. In all these models the statistical average over the ensemble of random lattices (the sum over all planar Feynman graphs with a fixed number of vertices) is regarded as the effect of the fluctuations of bidimensional quantum gravity. The critical coefficients of the statistical models are affected by gravity in agreement with the equations derived in the continuum approach [9] to conformal two-dimensional theories.

In the above-mentioned models, the critical coefficients were evaluated at a critical point, which is associated with the divergence of the perturbative series of planar Feynman graphs occurring, in general, for a negative value of the coupling constant that multiplies the highest monomial in the polynomial potential.

In the simplest one-matrix model the spectral density [10] is

$$u(\lambda)=\sqrt{4a^2-\lambda^2}f(\lambda) \qquad -2a\leqslant\lambda\leqslant 2a \tag{1}$$

where the polynomial $f(\lambda)$ is easily evaluated in terms of the potential and is non-negative on the compact support of $u(\lambda)$. The critical point mentioned above is characterised by the vanishing of $f(\lambda)$ at the border of the support, $\lambda=\pm 2a$, and it takes place for a negative value of the highest coupling constant. The spectral density then vanishes more rapidly than predicted by the Wigner semicircle law and the model exhibits a high-order phase transition. Several years ago [11-14] it was also proved that even the simplest matrix models exhibit a different phase transition: for finite negative values of lower-order coupling constants the polynomial $f(\lambda)$ vanishes for values of $\lambda$ inside the support. In this case the solution of the model is given by a spectral density whose support is the union of two or more segments. This splitting from one band to two (or more) bands, is somehow analogous to the conducting-dielectric transition.

In this letter we describe the phase diagram of the simplest one-matrix model where multicritical points may occur. We consider the partition function

$$Z(g_1, g_2, g_3) = \lim_{N\to\infty} C^{-1} \int \mathrm{d}^{N^2} M \exp\left[-\mathrm{Tr}\left(g_1 M^2 + \frac{g_2}{N} M^4 + \frac{g_3}{N^2} M^6\right)\right] \tag{2}$$

where

$$C = \int \mathrm{d}^{N^2} M \exp-(g_1|\mathrm{Tr}\, M^2) \tag{3}$$

and $M$ is a Hermitian $N \times N$ matrix. Using a well known procedure [10], one obtains a singular integral equation for the spectral density

$$\fint \mathrm{d}\mu \frac{u(\mu)}{\lambda - \mu} = g_1\lambda + 2g_2\lambda^3 + 3g_3\lambda^5 \tag{4}$$

which is solved, in the region of positive couplings, by

$$u_{\mathrm{I}}(\lambda) = \frac{1}{\pi}\sqrt{4a^2 - \lambda^2}\,[3g_3\lambda^4 + 2(g_2 + 3g_3a^2)\lambda^2 + (g_1 + 4g_2a^2 + 18g_3a^4)]. \tag{5}$$

The support of the spectral density is the segment $-2a \leqslant \lambda \leqslant 2a$ where $a^2$ is the unique positive solution of the cubic equation

$$60g_3a^6 + 12g_2a^4 + 2g_1a^2 - 1 = 0. \tag{6}$$

The correlation functions are easily evaluated as moments of the spectral density, and the free energy is:

$$E = -\tfrac{6}{5}g_2^2a^8 - \tfrac{6}{5}g_1g_2a^6 + (\tfrac{8}{5}g_2 - \tfrac{1}{3}g_1^2)a^4 + \tfrac{7}{6}g_1a^2 - \tfrac{1}{2} - \tfrac{1}{2}\log(2g_1a^2). \tag{7}$$

One may call this solution the perturbative solution, since its power expansion in the couplings reproduces the planar Feynman graphs of the model.

We now consider the analytic continuation of the perturbative solution to real non-positive values of $g_1$ and $g_2$ while keeping $g_3 > 0$. This continuation is not acceptable as a solution of the model if the spectral density is not positive definite on its support. This requirement is fulfilled in region I of the phase diagram in figure 1. The line $\gamma_1$ in figure 1 is determined by the condition $u_{\mathrm{I}}(\lambda = 0) = 0$, i.e.

$$g_1 + 4g_2a^2 + 18g_3a^4 = 0 \tag{8}$$

with $a^2$ given by (6).

In the region II of figure 1 we obtain a non-perturbative solution $u_{\mathrm{II}}(\lambda)$ of (4) with support $[-B, -A] \cup [A, B]$:

$$u_{\mathrm{II}}(\lambda) = \frac{1}{\pi}[(B^2 - \lambda^2)(\lambda^2 - A^2)]^{1/2}|\lambda|[3g_3\lambda^2 + \tfrac{3}{2}g_3(A^2 + B^2) + 2g^2]. \tag{9}$$

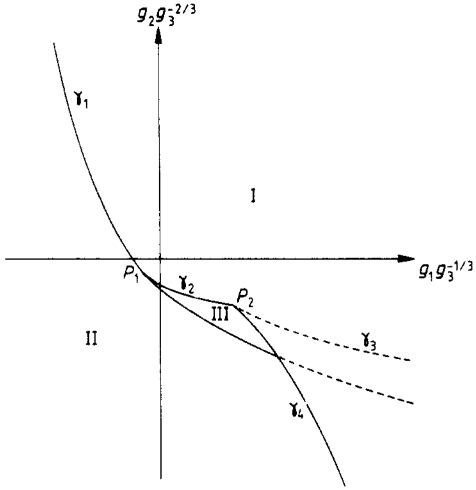


**Figure 1.** The phase diagram of the model in adimensional variables: I is the perturbative region, with solution $u_{\mathrm{I}}(\lambda)$ on a single segment; II is the domain for the two-segment solution $u_{\mathrm{II}}(\lambda)$; III is the domain for the three-segment solution. In the lower right region, bounded by $\gamma_1$ and $\gamma_4$, both $u_{\mathrm{I}}$ and $u_{\mathrm{II}}$ solve the saddle-point equation.

The square root has the arithmetic (positive) determination, and the extrema of the support solve the equations:

$$\begin{cases} g_1 + g_2(A^2+B^2) + \frac{3}{4}g_3[(A^2+B^2)^2 + \frac{1}{2}(A^2-B^2)^2] = 0 \\ \frac{3}{8}g_3(B^2-A^2)^2(B^2+A^2) + \frac{1}{4}g_2(B^2-A^2)^2 - 1 = 0. \end{cases} \tag{10}$$

By defining $y = (A^2+B^2) + \frac{2}{3}g_2/g_3$, one has

$$\begin{cases} y^3 + \frac{4}{3}y[g_1/g_3 - \frac{1}{3}(g_2/g_3)^2] + 4/(3g_3) = 0 \\ (B^2-A^2)^2 = 8/(3g_3 y). \end{cases} \tag{11}$$

The cubic equation has two positive roots if

$$\frac{1}{3}\left(\frac{g_2}{g_3}\right)^2 - \frac{g_1}{g_3} - \left(\frac{9}{4g_3}\right)^{2/3} \geqslant 0 \tag{12}$$

but only the largest one provides an acceptable solution of (10) in the region II of figure 1, corresponding to real values $B > A \geqslant 0$. Furthermore, this solution is continuous in the parameters in the same region of the parameter space. One can also see that the requirement $u_{\mathrm{II}}(\lambda) \geqslant 0$ on its support coincides with (12).

The condition that the two segments of the support merge into a single one, that is, $A = 0$, is easily seen to reproduce (6) and (8) by identifying $B^2 = 4a^2$. Therefore we conclude that the line $\gamma_1$ is the natural boundary for the non-perturbative solution $u_{\mathrm{II}}(\lambda)$.

We now return to the determination of the region of positivity for the perturbative solution $u_{\mathrm{I}}(\lambda)$. A new line $\gamma_2$ arises from the conditions $u_{\mathrm{I}}(\lambda_0) = u_{\mathrm{I}}'(\lambda_0) = 0$ with $0 < \lambda_0 < 2a$. This is given by (6) coupled to

$$15g_3a^4 + 2g_2a^3 + g_1 = \frac{1}{3}\frac{g_2^2}{g_3} \tag{13}$$

with

$$a^2 < -\frac{1}{3}\frac{g_2}{g_3} < 5a^2. \tag{14}$$

The two lines $\gamma_1$ and $\gamma_2$ intersect in $P_1$ with coordinates

$$g_1 g_3^{-1/3} = -(3/2)^{1/3} \qquad g_2 g_3^{-2/3} = -(3/2)^{2/3} \tag{15}$$

which also belongs to the parabola (12) and corresponds to the equality in the LHS of (14).

Since the perturbative solution $u_{\mathrm{I}}(\lambda)$ develops a zero in $\lambda_0$, $0<\lambda_0<2a$, on the line $\gamma_2$, we expect that below $\gamma_2$ the model has a non-perturbative solution $u_{\mathrm{III}}(\lambda)$ with support on three segments.

We then examine the possibility that $(4a^2-\lambda^2)^{-1/2}u_{\mathrm{I}}(\lambda)=0$ at $\lambda_0=2a$, that is

$$90g_3a^4+12g_2a^2+g_1=0. \tag{16}$$

Equations (16) and (6) define the lines $\gamma_3$ and $\gamma_4$ in figure 1. These lines originate at the point $P_2$ with coordinates

$$g_1 g_3^{-1/3} = \tfrac{3}{2}(60)^{1/3} \qquad g_2 g_3^{-2/3} = -\tfrac{1}{2}(450)^{1/3}. \tag{17}$$

In the region bounded by $\gamma_3$ and $\gamma_4$ the cubic equation (6) has three positive roots, with two roots coinciding on the boundary ($0<a_1^2<a_2^2=a_3^2$ on $\gamma_3$, $0<a_1^2=a_2^2<a_3^2$ on $\gamma_4$). Outside this region the cubic equation (6) has only one positive root. We find that the continuation of the unique root of the cubic equation from region I across $\gamma_3$ coincides with the smallest root $a_1^2$ in the whole region bounded by $\gamma_3$ and $\gamma_4$. Equation (16) is solved by the largest root $a_3^2$ on $\gamma_3$ and by the smallest $a_1^2$ on $\gamma_4$. Equation (8) is solved inside this region by the root $a_3^2$. We conclude that the perturbative solution $u_{\mathrm{I}}(\lambda)$ may be continued across $\gamma_3$ in the whole region up to the boundary $\gamma_4$. In the infinite region bounded by $\gamma_1$ and $\gamma_4$ both solutions $u_{\mathrm{I}}(\lambda)$ and $u_{\mathrm{II}}(\lambda)$ exist, the physical one being that with lower free energy.

By inserting the value $a^2=-\frac{1}{15}g_2/g_3$ that saturates the RHS of inequality (14) into (13) we obtain

$$g_1g_3=\tfrac{2}{5}g_2^2. \tag{18}$$

The only critical point belonging to this parabola is the multicritical point $P_2$.

While this work was being completed, several interesting articles were written [15-17] which discuss matrix models where at certain multicritical points a special rescaling is performed, allowing all terms in the topological expansion to have the same relevance†. This possibility is particularly interesting for the non-perturbative understanding of the superstring theory in non-critical dimension. In [15-17] the matrix models are analysed by the technique of orthogonal polynomials which is equivalent to the saddle-point method of this letter for the leading term in the topological expansion, and it is better suited to the non-leading terms. In the perturbative region I, our results agree with the analysis of [15] and if we rewrite our couplings $g_i \to g_i/\lambda$ to agree with the notation of that work, (6) of this letter coincides with their equation

$$\omega(R)=2g_1R+12g_2R^2+60g_3R^3=\lambda \tag{19}$$

which, together with (18), identifies the multicritical point $P_2$.

† We thank D Zanon for providing us with copies of these preprints.

In the non-perturbative regions II and III the technique of orthogonal polynomials requires modification: as it was shown in [18], different sets of polynomials must be interpolated by distinct functions. We are completing this analysis for region II which will be published shortly.

Finally, we remark that in the limit $g_3 \to 0^+$ the phase diagram in figure 1 recovers the previously known critical points of the quartic model. The point at infinity along the line $\gamma_1$ corresponds to $g_1 = -2\sqrt{g_2}$ which marked the transition discussed in [11, 12]. The point at infinity along the line $\gamma_4$ corresponds to $g_2 = -g_1^2/12$ which was identified in [10] with the radius of convergence of the perturbative expansion of the planar theory and it was later interpreted as a critical point in statistical mechanics models on random graphs. By adding the sextic term to the quartic interaction, as in the present letter, this singular point may be better understood as a critical phase transition point†.

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